Referee Report

Manuscript Title: *Construction of a New Hypersurface Family Using Spherical Product in Minkowski Geometry*
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The manuscript introduces a new family of hypersurfaces generated through the **spherical product of planar curves** in **Minkowski 4-space** $E_1^4$and generalizes this concept to n-dimensional Lorentzian spaces. The authors provide explicit parameterizations, derive the **Gaussian and mean curvatures**, and present **conditions for flatness and minimality**. The paper also interprets several **hyperquadrics** (superellipsoids, superhyperboloids, toroids) as special cases of spherical product hypersurfaces, supported by graphical visualizations.

General Evaluation

The topic belongs to the area of **differential geometry of hypersurfaces in semi-Riemannian spaces**, which is suitable for *Symmetry*. The construction is correctly formulated, and the algebraic derivations appear consistent. The extension of the spherical product to Minkowski geometry is logically presented, and the paper's structure (Introduction, Basic Notations, Main Construction, Examples, Generalization, Conclusion) follows the journal's format.

However, the **scientific novelty is modest**. The main results—flatness corresponding to one generating curve being straight, and minimality under specific conditions—are **direct analogues** of known Euclidean results (e.g., Bulca & Arslan, *An. St. Univ. Ovidius Constanța*, 2012; Büyükkütük & Öztürk, *Turk. J. Math.*, 2024). The transition to Minkowski space changes only the signature of the metric, without introducing new geometric phenomena or invariant formulations. Consequently, the paper extends known constructions rather than developing genuinely new theory.

Mathematical and Presentation Issues

- The formulas for the **first and second fundamental forms** and curvature functions are correct but presented in overly long computational form; a tensorial or invariant approach would improve clarity.
- The **English language** requires significant editing. Common mistakes include missing articles, incorrect verb tenses, and misspellings ("standart embedding", "hypersurfaces' projection", "superhyperbola").
- The **figures** are illustrative but lack scale or axis labeling; captions could explain geometric meaning rather than software details.
- Several **references** are self-citations or overlap with closely related earlier works; inclusion of broader sources (e.g., O'Neill, Lopez, Magid) is recommended.

Strengths

- Clear structure and readable organization.
- Complete derivations of Gaussian and mean curvatures.
- Concrete geometric examples connecting spherical products to hyperquadrics.

Weaknesses

- Limited originality; mostly computational extension of known Euclidean constructions.
- Lack of geometric or physical interpretation.
- Language and stylistic issues throughout.

Recommendation

The paper is **mathematically correct** and within the journal's scope but needs **major language and stylistic revision** and a more explicit discussion of its **geometric contribution** beyond prior literature.

Decision: *Major Revision.*

standart embedding	**standard embedding**
a standart embedding from n−sphere into En+1	**a standard embedding of the n-sphere into** E^{n+1}
The product α(u) ⊗ β(v) results in a higher dimensional embedding...	**The product α(u) ⊗ β(v) results in a higher-dimensional embedding...**
They defined the spherical product α(u)⊗β(v) of two planar curves as...	**They defined the spherical product α(u) ⊗ β(v) of two planar curves as follows:**
is a standart embedding from n−sphere into En+1	**is the standard embedding of the n-sphere into** E^{n+1}
superellipses are the special cases of Lame curves that are called after Gabriel Lame	**Superellipses are special cases of Lamé curves, named after Gabriel Lamé.**
The spherical product is taken into consideration on hypersurfaces for the first time...	**The spherical product is considered on hypersurfaces for the first time...**
the related hypersurfaces' projection in 3−dimensional form can be plotted by using Maple	**the corresponding 3D projection can be plotted using Maple**
we get the result	**we obtain the result**
we can see from (3.1), the spherical product β(v)⊗γ(w) corresponds to...	**As seen from (3.1), the spherical product β(v) ⊗ γ(w) corresponds to...**

This parametrization corresponds to a toroidal hypersurface in E4, and satisfies...	**This parametrization corresponds to a toroidal hypersurface in $E^4$and satisfies...**
This corresponds to an hyperellipsoid...	**This corresponds to a hyperellipsoid...**
which has the following parametrization	**which has the following parametrization:**
let the three curves α, β, γ be chosen as...	**Let the three curves α, β, and γ be chosen as...**
the spherical product hypersurface generated by these curves has the following parametrization	**The spherical product hypersurface generated by these curves is parametrized by:**
the following formal definition is provided:	**The following formal definition is given:**
has the parametrization as	**has the parametrization**
the Minkowski 4−space E4 1 represents a four dimensional real vector space R4 equipped with the metric tensor...	**The four-dimensional Minkowski space $E_1^4$is a real vector space $\mathbb{R}^4$equipped with the metric tensor...**
In particular, choosing c1 = 2, c2 = −1, c3 = c4 = 3, and w = π/2, the related hypersurface s projection...	**In particular, choosing ..., the corresponding hypersurface’s projection...**
The projection into three-dimensional space is carried out by eliminating the fourth component...	**The projection onto three-dimensional space is obtained by eliminating the fourth component...**
Then, a spherical product hypersurface in En1 is defined by C1 ⊗ C2 ⊗ ...	**Then, a spherical product hypersurface in $E_1^n$is defined as $C_1 \otimes C_2 \otimes$...**
The matrices corresponding to the first fundamental form I and the second fundamental form I I are respectively given by...	**The matrices corresponding to the first fundamental form $I$and the second fundamental form $II$are respectively given by...**