RESPONSE TO THE REVIEWER 3

Abstract: "The paper by Yang Yue et al. is devoted to the theoretical study of exceptional points (EPs) in the coupled-waveguide non-Hermitian system. The refractive index of the waveguides is modulated, so that the real and imaginary parts change out of phase. One of the waveguides contains material with loss, whereas another is characterized by gain of the same magnitude. The authors find the positions of EPs in the parameter space for both the Hamiltonian and the scattering matrix. Then the dependence of wave transmission on the waveguide length is calculated and analyzed for different situations: at EPs and out of EPs. The results seem to be reliable and worth to be published. However, I have a feeling that the analysis is not full and some important aspects were not discussed in the manuscript as described in my further comments."

Our reply: We are grateful to the reviewer for considering our results reliable and worthy of publication. We also wholeheartedly appreciate the thoughtful suggestions the reviewer has offered to help us enhance our manuscript. We will respond carefully to each of the reviewer's comments.

Comment 1: "The authors correctly note that the EPs for the Hamiltonian and the scattering matrix are generally different. However, it is well known that the scattering matrix can be defined in different ways (the components of E_{in} and E_{out} can be ordered differently). This ambiguity will generally influence the position of EPs. See, e.g., Phys. Rev. A 101 (4), 043834 (DOI 10.1103/PhysRevA.101.043834). It is possible that, for another choice of the scattering matrix, the EPs will be the same as for the Hamiltonian. I think that this possibility is worth to be studied in the paper."

Response 1: We are grateful for the constructive suggestion put forward by the reviewer, and we have carefully read and discussed the work mentioned by the reviewer. The scattering matrix does indeed have different definitions depending on the order of the components of E_{out} when the order of the components of E_{in} remains unchanged. The scattering matrix we used in our manuscript, similar to $\hat{S}_2$ mentioned in Phys. Rev. A 101, 043834 (2020), can accurately predict the EPs where the PT-symmetry gets broken.

In our manuscript, the difference of the EPs between the Hamiltonian H_{d} and the scattering matrix is not caused by the different definitions of the scattering matrix. This is because the definition of the Hamiltonian in our manuscript is different from that in Phys. Rev. A 101, 043834 (2020). In our manuscript, the Hamiltonian H_{d} satisfies

$$\frac{\mathrm{d}}{\mathrm{d}z}\Psi(z) = -iH_{\mathrm{d}}\Psi(z). \tag{0.1}$$

The transmission matrix of the system is $T = e^{-iH_{\mathrm{d}}z}$. The transmission matrix T has the same EP as the Hamiltonian H_{d}. In the Phys. Rev. A 101, 043834 (2020), the Hamiltonian $\hat{H}$ satisfies

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix} H(z,t) \\ E(z,t) \end{pmatrix} = \hat{H}\begin{pmatrix} H(z,t) \\ E(z,t) \end{pmatrix}. \tag{0.2}$$

The scattering matrix of the system is $S = e^{-i\hat{H}t}$. Thus the scattering matrix S has the same EP as the Hamiltonian $\hat{H}$. To ensure scientific, we added "Here, this scattering matrix S is similar to the one mentioned in Ref. [42] that has the same EP as the Hamiltonian in Ref. [42]." after Eq. (9) in the revised manuscript to explain the correlation between our scattering matrix and the one in this article, here Ref.[42] is Phys. Rev. A 101, 043834 (2020).

We are truly appreciative of the suggestions put forth by the reviewer regarding the exploration of different scattering matrices. After careful consideration, we have decided to reserve this aspect for future research endeavors. Our decision is based on two main factors:

1. We currently possess a limited understanding of the physical significance of EPs within the context of various scattering matrices. Further investigation and comprehension are necessary before we can adequately delve into this topic.

2. In the case of a double waveguide system, the output electric field E_{out} comprises four components. This complexity leads to the existence of 24 distinct definitions for scattering matrices. Within these numerous definitions, there likely reside a wealth of intricate physical phenomena. Given the scope and focus of our current manuscript, it would be more appropriate to explore these rich phenomena in subsequent studies rather than incorporating them here.

Comment 2: "The choice of the structure is not discussed in detail in the paper. In particular, it is not stated directly that the waveguides are with loss and gain, respectively. This is the physical reason for transmission growth shown in the paper. Moreover, the choice of modulation ($\Delta n_1 = -\Delta n_2$) is such that the coupled waveguides do not obey the popular symmetries (such as PT or anti-PT symmetry). What is the reason for such a choice? The discussion of the structure symmetries and their influence will be very relevant for the paper in the journal called "Symmetry"."

Response 2: We'd like to extend our heartfelt thanks to the reviewer for the constructive comment. We opted for this particular modulation scheme primarily due to the following reasons: 1. We investigate the transmission and reflection phenomena of double waveguides under the condition that the modulated refractive index exhibits a certain symmetry along the waveguide arrangement direction. For example, even symmetry corresponds to $\Delta n_1(z) = \Delta n_2(z)$, and odd symmetry corresponds to $\Delta n_1(z) = -\Delta n_2(z)$. 2. The modulation scheme $\Delta n_1(z) = \Delta n_2(z)$ with even symmetry in the waveguide arrangement direction will not generate an EP with three defect states. Therefore, we choose $\Delta n_1(z) = -\Delta n_2(z)$ as the modulation scheme used in our manuscript.

Following the reviewer's suggestion, we modified the description of the modulation scheme in the opening paragraph of Section 2.2 of the revised manuscript to explain the reasons for choosing this modulation scheme and clearly pointed out that this would cause the two waveguides to have the characteristics of loss and gain respectively, resulting in an increase in the total energy output of the system.

Comment 3: "The situations at EPs and out of EPs are considered in the paper, but it is not clear whether EPs provide any qualitative differences in the system response. I can only see such qualitatively different behavior in Fig. 3 where the linear dependence of transmission on length is demonstrated. What about other cases? Is it possible that the growth asymptotics or some other parameters radically change at the EPs? Such details will be very helpful to fully understand the uniqueness of EPs for the system's response."

Response 3: We are so very grateful for the reviewer's truly constructive comment. If we understand correctly, the reviewer's question is about the unique behaviors at various types of EPs, which we clarify in detail below.

1. CS2-1_{H}($CS3_{\mathrm{S}}$): As noted by the reviewer, the output of the scattering matrix exhibit three scenarios: 0, proportional to L, and proportional to L^2, where L is the length of the waveguide. In our manuscript, the output value of 0 manifests in the system as no reflection occurring when light waves are incident from the left side of the waveguide.

2. CS2-1_{S}: The distinctiveness resides in the existence of a scenario wherein the output of the scattering matrix is zero. In our manuscript, the output value of 0 manifests in the system as no reflection occurring when light waves are incident from the right side of the waveguide. In the regions II and III of Section 3.5, there will be no situation where the scattering matrix output is 0

3. CS2-2_S: since the denominator of the scattering matrix is $\cos^2(\sqrt{a}L)$, there is only one divergence point in the interval $L \in [0, \pi/\sqrt{a}]$. In the regions IV of Section 3.5, the denominator of the scattering matrix is $1+C_qC_{-q}\sin^2(\sqrt{a}L)/a$, where $C_qC_{-q} < 0$, $a > 0$ and $|C_qC_{-q}| > |a|$. In this case, there are two divergence points in the interval $L \in [0, \pi/\sqrt{a}]$.

4. CS2-2_H: At this EP, the uniqueness lies in the fact that the transmitted light is directly proportional to $1/L$ when $L \to \infty$. In the regions I and V of Section 3.5, $a < 0$, the transmitted light is directly proportional to $e^{-\sqrt{|a|}L}$ when $L \to \infty$.

We hope these detailed introductions can effectively address the reviewer's question.

Comment 4: "In order to fully understand the difference between different cases, it would be great to show the refractive index profiles for every situation. Then, the reader will clearly see the difference between the structures studied."

Response 4: We appreciate the reviewer's suggestion regarding the inclusion of diagrams illustrating refractive index distributions under various conditions. After careful consideration, we have decided to retain the current form of the manuscript without adding those figures. Our reasoning is that, since the only parameter varied in the modulated refractive index $\Delta n(z) = m\cos(qz) + i[g + p\sin(qz)]$ is p, the resulting refractive index distributions do not exhibit visually substantive variations across different values of p. Therefore, we believe that including such figures would not substantially enhance the clarity of our manuscript.

To ensure transparency, we have revised the opening paragraph in Section 3 of the revised manuscript, and we modified "In this section, we examine light transmission across distinct regimes." to "In this section, we will generate different EPs by adjusting the parameter p and investigate the optical transmission phenomena across different parameter spaces.". We are confident that this textual clarification adequately conveys the relevant information to the reader.

Comment 5: "In section 3.2, we read that "if the wave is incident from the right side of the system, it will be fully transmitted and not reflected". However, I see in Fig. 4 that the radiation is not just "fully transmitted", but can be significantly amplified. Therefore, it is interesting to analyze not only transmissions and reflections separately, but the total energy output as well."

Response 5: We are grateful for the suggestion provided by the reviewer. Following the reviewer's suggestion, the description of "fully transmitted" has been removed from the revised manuscript and we added the analyses of total energy output following each transmission and reflection analysis in the revised manuscript. Next, we will describe the variation of the total energy output in EP of various orders and in different regions divided by EPs:

1. At CS2-1_H (CS3_S), the total energy output exhibit two scenarios: proportional to L^2 and L^4 when $L \to \infty$, where L is the length of the waveguide.

2. At CS2-1_S, the total energy output will oscillate periodically, with a period of $\pi/\sqrt{a}$.

3. At CS2-2_S, the total energy output will oscillate periodically with a period of $\pi/\sqrt{a}$ and the divergent responses will occur when $\sqrt{a}L = \pi/2 + n\pi$ (here n is a natural number) if in an ideal system.

4. At CS2-2_H, the total energy output will exhibit a divergent response when $1+(C_0^2+t^2)L^2 = 0$ if in an ideal system, and then it will approach a value when $L \to \infty$.

5. At regions divided by EPs, the total energy output is $\varepsilon = [|d(L)|^2+|d(L)e(L)|^2+|f(L)|^2+|e(L)f(L)|^2]/[b(L)f(L)-d^2(L)]^2$. For I and V, the total energy output will rapidly increase and then tend to a value. For II, III and IV, the

total energy output will oscillate periodically, and for IV, in addition to this, there will also be periodic divergent responses.

Comment 6: "In section 3.3 and further, the authors find the situation when "the outgoing amplitude tends to infinity". What does this mean physically? As a rule, infinity implies that consideration has some limitations and does not take into account important factors. In particular, here we see that the wave energy goes to infinity at a finite distance. Where does this infinite energy come from? I think that the reason is that gain is assumed to be nondepleted that will be obviously incorrect for very strong waves. A brief discussion of the limitations of the model used is needed in the paper."

Response 6: We thank the reviewer for raising this good point. In our manuscript, the divergence point appears where the denominator of the scattering matrix is zero, i.e., $bf - d^2 = 0$ [see Eq. 17 of the revised manuscript]. Physically, it implies the potential existence of a bound state or a quasi-bound state [see e.g. Phys. Rev. B 60, 15732 (1999)]. The study of such bound states is beyond the scope of this manuscript, which is in fact the subject of our ongoing project. Therefore, we choose not to include this in our current manuscript. Following the reviewer's suggestion, we added a discussion of the limitation of the model in Section 3.3 of the revised manuscript. We hope this addition makes our work more comprehensive and robust.

SUMMARY OF MAJOR REVISIONS

In the revised manuscript, changes are marked by blue color.

1. We rewrote the abstract and the introduction.

2. In Section 2.1, we briefly outlined the derivation process of Eqs. (5) and (6).

3. In Section 2.1, we have added statements about the approximations used in the manuscript.

4. In Section 2.1, we have added the derivation of the single waveguide scattering matrix.

5. In Section 2.1, we have added an introduction of the finite element method and Fig. 1, which presents a numerical comparison between the finite element method and the coupled mode method.

6. In Section 2.2, we have added the reasons for choosing refractive index modulation $n_1(z) = -n_2(z)$.

7. We have corrected the incorrect description of the modulated refractive index in Fig. 2 (Fig. 1 in the original manuscript).

8. We have added the discussion on the physical significance of the eigenvalue splitting of Hamiltonian.

9. In Section 2.2, we have added the discussion on the cases where the coupling coefficient t between waveguides tends to 0 and ∞.

10. We have added Table 1 to introduce the symbols used in the manuscript.

11. In Section 3, we have added the discussion on the total energy output.

12. In Section 3.3, we have added the discussion on the restrictions on the occurrence of divergence points.

13. We have added Table 2 to summarize the key behaviors in different regions.

14. In Section 4, we have added the discussion on the feasibility of the experiment and future research.

15. We removed six references Refs. [12-14], [16], [18-19] from the original manuscript and added thirteen references Refs. [23-25], [27-31], [34], [39], [42].