# RESPONSE TO THE REVIEWER 2

**Abstract: "The work presents an innovative exploration of exceptional points (EPs) in a non-Hermitian coupled waveguide system, demonstrating unique and known optical phenomena such as unidirectional transmission, amplification, periodic oscillations, and divergences through Hamiltonian and scattering matrix analyses. While the mathematical modeling and numerical results are contribute meaningfully to non-Hermitian photonics, refinements in validation, visualization, and discussion of practical implications would strengthen the manuscript by addressing these comments."**

**Our reply**: We sincerely appreciate the reviewer's thoughtful evaluation of our manuscript, recognizing its value and significance. We are also deeply grateful for the constructive suggestions they provided to help us further improve it. We will address each comment with great care and consideration.

**Comment 1: "Expand the abstract and introduction to more explicitly highlight the novelty of extending EP engineering beyond PT-symmetric constraints, and incorporate a brief quantitative comparison (e.g., via a table) with prior waveguide studies (refs [37] and others) to underscore how your non-PT-symmetric approach enables broader phenomena like the observed divergences."**

**Response 1**: We truly appreciate the reviewer's constructive comment. Following the suggestion of the reviewer, we have carefully rewritten both the abstract and the introduction. In the introduction of our revised manuscript, we have deliberately structured the content to first introduce the EP, followed by a review of previous studies on PT symmetry systems. This arrangement is intended to more clearly highlight that our system uniquely incorporates EP beyond the scope of PT symmetry, thereby giving rise to novel phenomena worth exploring.

**Comment 2: "Clarify the refractive index modulation in Section 2.1 by providing explicit derivations or numerical examples for the integrals in Eqs. (5) and (6), and validate these against alternative methods (e.g., finite element simulations) to confirm the accuracy of the coupled-mode approximations under the assumption $n_0 \gg \Delta n(z)$."**

**Response 2**: We truly appreciate the reviewer's constructive and thoughtful comment. In response, we have made revisions to Section 2.1 of the manuscript. Specifically, we have provided a concise overview of the derivation process for the two integrals.

We also compared the numerical results of the finite element method and the coupled mode method in the single-waveguide model, as shown in the Fig. A. Both methods have the same results. In the revised manuscript, we have added an introduction of the finite element method and Fig. 1, which presents a numerical comparison between the finite element method and the coupled mode method. This addition aims to clearly illustrate the accuracy of the coupled mode method under the condition where $n_0 \gg \Delta n(z)$.

**Comment 3: "In Section 2.2, derive the scattering matrix (Eq. 12) more step-by-step, including intermediate expressions for the transmission matrix T(z), and analyze the stability of the eigenvalues (Eq. 13) by plotting their trajectories in the complex plane for varying parameters beyond just p (e.g., include variations in m or g) to better illustrate EP coalescence."**

**Response 3**: We truly appreciate the reviewer's constructive comment. In response to the feedback, we have incorporated a brief derivation of the scattering matrix for a single waveguide in Section 2.1 of the revised manuscript. We believe this addition effectively clarifies the process through which we derived the scattering matrix.

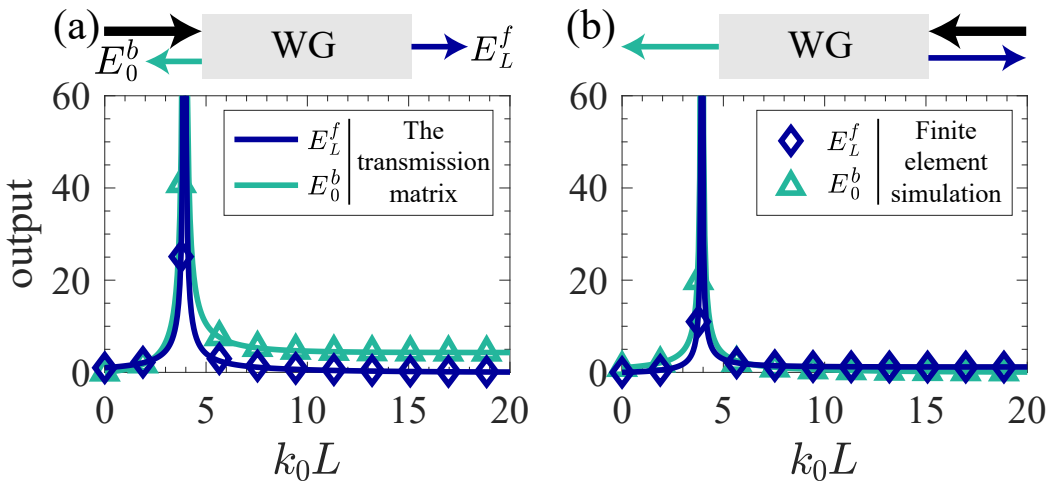


FIG. A. The outgoing amplitudes as a function of waveguide length $L$, numerically calculated via transmission matrix and finite element simulations respectively, when wave is incident from the (a) left and (b) right sides of the waveguide. Here $p = 3.5$, $m = -4/\pi$, $g = 1$, $n_0 = 10$. The blue represents the wave emitted from the right side of the waveguide, while the green represents the wave emitted form the left side. The solid line represents amplitude calculated via the transmission matrix, while the diamond and triangular denote amplitude computed through finite element simulation.

Regarding the suggestion to plot the trajectory of eigenvalues in the complex plane, we respectfully hold a different viewpoint. Our main emphasis in this study is on the EP generated by varying the parameter $p$. Therefore, we feel that plotting eigenvalues as a function of other parameters would not serve a practical purpose and could potentially overcomplicate the visualization. Moreover, we are confident that the current visualization sufficiently illustrates the EP coalescence phenomenon.

**Comment 4: "Enhance Figure 2 by adding annotations for the exact EP locations and parameter values on the plots, and discuss the physical implications of the eigenvalue splitting/degeneracy (e.g., relating imaginary parts to gain/loss) in the context of potential experimental realizations, such as in polariton systems mentioned in the abstract."**

**Response 4**: We are sincerely grateful to the reviewer for the constructive comment. In the revised version of the manuscript, we have incorporated the expression for the eigenvalue split at the EP into the caption. Additionally, we have carefully discussed the physical implications of this eigenvalue splitting within the context of the experimental background in Section 2.2.

**Comment 5: "For the transmission analyses in Section 3, incorporate analytical expressions for the output amplitudes as functions of L (beyond numerical plots) in key cases like CS2-1H (Figure 3), and quantify the amplification rates or oscillation periods using asymptotic approximations to provide deeper insights into scalability for device applications."**

**Response 5**: We are truly appreciative of the reviewer's constructive comment. In Section 3 of the revised manuscript, we have carefully provided explicit expressions for the scattering matrix in each distinct scenario. By integrating these expressions with the definitions of the symbols used in the scattering matrix from Section 2, one can effortlessly derive functions that describe how the output changes with respect to the length parameter $L$. To enhance clarity, we have also added a note in Sections 3.3 and 3.4, explicitly stating that the definitions of these symbols can be found in Section 2.

**Comment 6: "Address potential numerical artifacts in divergent responses (e.g., Figures 5 and 6) by specifying the simulation methods (e.g., discretization schemes or boundary conditions) used for amplitude calculations, and perform a sensitivity analysis on parameters like t or q to evaluate robustness against fabrication imperfections."**

**Response 6**: We are truly grateful to the reviewer for the constructive comment. We would like to clarify that the divergence point observed in our study stems from the fact that we are analyzing an idealized system where the gain remains constant and does not diminish as the electric field intensifies. This phenomenon is not a result of numerical inaccuracies. In light of the reviewer's valuable suggestion, we have incorporated a discussion on this aspect in Section 3.3 of the revised manuscript.

Furthermore, throughout Section 3 of the revised manuscript, we have consistently utilized scattering matrices for our computations and provided precise analytical expressions for these matrices. We are confident that this approach effectively illustrates how the output results vary with different parameters, and therefore, we believe that conducting additional sensitivity analysis at this stage would be redundant.

**Comment 7: "In Section 3.5, reorganize the discussion of regions I-V into a table summarizing key behaviors (e.g., periodicity, divergence conditions) for quick reference, and explore the phase transitions between regions by computing Berry phases or encircling dynamics around EPs to connect with broader non-Hermitian topology concepts."**

**Response 7**: We are truly grateful to the reviewer for the constructive comment. In accordance with the reviewer's helpful suggestion, we have incorporated a table in Section 3.5 of the revised version to succinctly summarize the key behaviors observed across the five regions.

We would like to express our sincere appreciation for the reviewer's insightful suggestion regarding exploring the phase transition between by computing Berry phase or encircling dynamics around EPs. We acknowledge that there indeed exists a topological aspect at the EP, which can be effectively captured via the Berry phase. However, in our manuscript, our primary research emphasis is centered on the transmission and reflection characteristics of waveguides across various parameter regions. Delving into topological aspects would extend beyond the intended scope of our study. In order to maintain the coherence and focus of the manuscript, we have made the careful decision not to incorporate any topology related content at this manuscript. However, we are eager to add topology related research in future studies to further explore this interesting field.

**Comment 8: "Strengthen the conclusion by discussing experimental feasibility (e.g., material choices for realizing $\Delta n(z)$ modulation, such as using electro-optic effects), and suggest extensions like incorporating nonlinearity (ref [19,20]) or multi-waveguide arrays to broaden the applicability to metamaterials or lasers."**

**Response 8**: We sincerely appreciate the reviewer's constructive comment. Acting upon the reviewer's suggestion, we have added a paragraph in the revised manuscript that describes the experimental protocol within our conclusion. Additionally, we have outlined potential avenues for future expansion to provide a clearer roadmap for subsequent research.

**Comment 9: "Throughout the manuscript, correct minor typographical issues (e.g., "perodically" to "periodically" in Figure 1 caption, "satisifies" to "satisfies" in Eq. 10 discussion) and provide a nomenclature section listing all symbols (e.g., $\sigma_x$, $\alpha$, a) for improved readability."**

**Response 9**: We thank the reviewer for the careful reading and we apologize for the typos. We have corrected the errors in the manuscript and added a table at the end of Section 2 that introduces all symbols used in the manuscript.

**Comment 10: "The introduction should be rewritten about many previous works with PT-symmetric couplers (like the current work) to address the readers about the potential and applications of non-Hermitian systems, for example: https://doi.org/10.1364/AO.58.002867"**

**Response 10**: We thank the reviewer for the constructive comments. Following their suggestions, in the revised manuscript, we have rewritten the introduction section and incorporated some findings on PT symmetry "Examples include: unidirectional reflectionless phenomena observed in optical non-Hermitian PT-symmetric systems on silicon-on-insulator platforms [32], nonreciprocal transmission resulting from PT-symmetry breaking in coupled resonators [33], asymmetric transmission achieved through integrated waveguide-based PT-symmetric optical circulators [34], and unidirectional transparency realized at exceptional points in non-Hermitian acoustics through the introduction of PT symmetry [31]." Here [34] is Appl. Opt. 58, 2867 (2019) mentioned by the reviewer.

## SUMMARY OF MAJOR REVISIONS

In the revised manuscript, changes are marked by blue color.

1. We rewrote the abstract and the introduction.

2. In Section 2.1, we briefly outlined the derivation process of Eqs. (5) and (6).

3. In Section 2.1, we have added statements about the approximations used in the manuscript.

4. In Section 2.1, we have added the derivation of the single waveguide scattering matrix.

5. In Section 2.1, we have added an introduction of the finite element method and Fig. 1, which presents a numerical comparison between the finite element method and the coupled mode method.

6. In Section 2.2, we have added the reasons for choosing refractive index modulation $n_1(z) = -n_2(z)$.

7. We have corrected the incorrect description of the modulated refractive index in Fig. 2 (Fig. 1 in the original manuscript).

8. We have added the discussion on the physical significance of the eigenvalue splitting of Hamiltonian.

9. In Section 2.2, we have added the discussion on the cases where the coupling coefficient $t$ between waveguides tends to 0 and $\infty$.

10. We have added Table 1 to introduce the symbols used in the manuscript.

11. In Section 3, we have added the discussion on the total energy output.

12. In Section 3.3, we have added the discussion on the restrictions on the occurrence of divergence points.

13. We have added Table 2 to summarize the key behaviors in different regions.

14. In Section 4, we have added the discussion on the feasibility of the experiment and future research.

15. We removed six references Refs. [12-14], [16], [18-19] from the original manuscript and added thirteen references Refs. [23-25], [27-31], [34], [39], [42].