# RESPONSE TO THE REVIEWER 1

**Abstract: "In this MS, through the design of the refractive index distribution in a coupled waveguide structure, the authors generate multiple EPs and observe a series of non-Hermitian optical signatures. They argued that this work involves EPs in both the Hamiltonian and the scattering matrices, and are of distinct types and emerge at distinct positions in the system.**

**I do not think this MS provides new insight into non-Hermitian optics, based on the comments listed below."**

**Our reply**: We sincerely thank reviewer for reading this paper and for providing valuable comments. The reviewer is critically concerned about the novelty of our manuscript, mainly based on the arguments: 1). Eps in similar structure has been discussed in some literatures, and 2) our theoretical calculation is based on a simplified scattering matrix approach. In the following, we address the reviewer's concerns point by point below.

**Comment 1: "EPs in similar structure has been discussed in some literatures, for example, Parity-Time Symmetry in Laterally Coupled Bragg Waveguides by Yann G. Boucher and Patrice Féron, IEEE JOURNAL OF QUANTUM ELECTRONICS, VOL. 55, NO. 6, DECEMBER 2019 6000209."**

**Response 1**: We sincerely appreciate the reviewer's comment. The EP physics discussed in our manuscript is crucially different from the work mentioned by the reviewer: while IEEE J. Quantum Electron. 55, 6000209 (2019) mainly concerns the second-order EP, our work focuses on the case of higher-order EP. Below, we will elaborate on these differences:

1. The differences of EP physics. In the article given by the reviewer, a system can always be perfectly reduced into two simpler subsystems. This is equivalent to two decoupled single-waveguide systems. In this case, it is impossible for a higher-order EP to exist with three defect states. Therefore, what they observe can only be an EP with a single defect state or two defect states. The phenomenon of higher-order EPs cannot be observed. In our model, we not only present the phenomena at EP with a single defect state and two defect states, but also show the phenomenon at EP with three defect states.

2. The difference in symmetry. In the article given by the reviewer, what they considered was a PT-symmetric system. In our manuscript, we are considering a non-PT-symmetric system.

3. The differences in research content. In the paper IEEE J. Quantum Electron. 55, 6000209 (2019), the main focus is on the gain requirements for achieving laser oscillation. In our manuscript, we mainly focus on the differences in optical transmission phenomena at EP of various orders and find that EP divides the parameter space into different regions, each of which has unique optical transmission phenomena.

Based on the above points, we have clarified that our work is fundamentally different. Therefore, we believe that our research will provide new insights for non-Hermitian optics.

At the meantime, we are grateful to the reviewer for pointing our attention to this article. In the revised manuscript, we have cited it in "While most current research on EPs remains focused on PT-symmetric architectures [26, 35-39] and second-order EPs, the concept has subsequently been generalized to more general non-PT-symmetric systems, where chiral transmission of eigenmodes has been successfully demonstrated [40]." which is in the second paragraph of the introduction of the revised manuscript, where Ref.[39] is IEEE J. Quantum Electron. 55, 6000209 (2019).

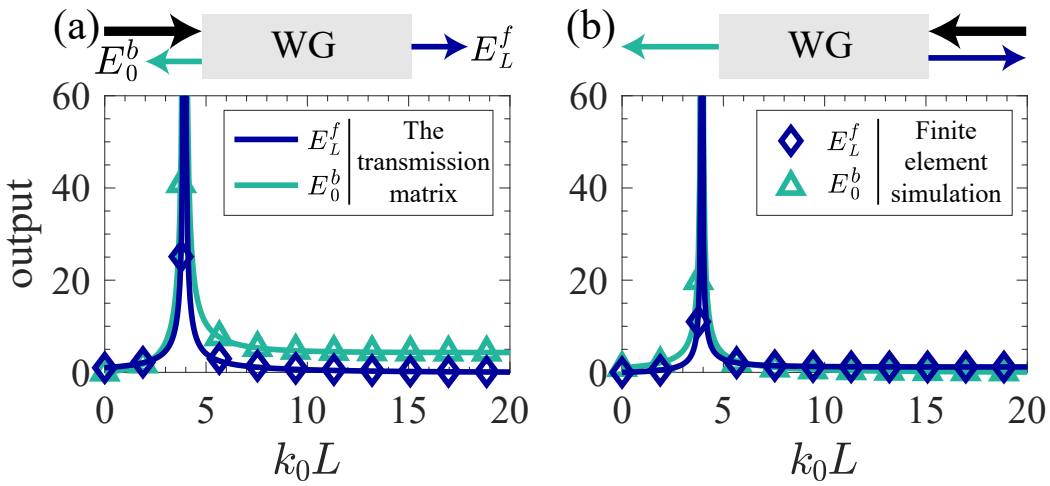


FIG. A. The outgoing amplitudes as a function of waveguide length $L$, numerically calculated via transmission matrix and finite element simulations respectively, when wave is incident from the (a) left and (b) right sides of the waveguide. Here $p = 3.5$, $m = -4/\pi$, $g = 1$, $n_0 = 10$. The blue represents the wave emitted from the right side of the waveguide, while the green represents the wave emitted form the left side. The solid line represents amplitude calculated via the transmission matrix, while the diamond and triangular denote amplitude computed through finite element simulation.

**Comment 2: "The whole MS is based on a simple scattering matrix. But according to the states of the art advances in non-Hermitian optics, this simple model is not enough. The authors should provide more information in supporting the results, especially by using numerical ones (such as FEM and FDTD) that shows the field distributions inside the structures."**

**Response 2**: We sincerely appreciate the thoughtful comment from the reviewer. The reviewer has raised a valid point regarding the sufficiency of relying solely on the scattering matrix in our manuscript and has suggested incorporating the field distribution inside the structure.

The aim of our manuscript is the study of light transport at various type of EPs in the double waveguide system, in particular higher-order EPs, which have remained largely explored so far. To extract and demonstrate the essential physics, therefore, we choose a theoretical approach based on the scattering matrix, building upon experimental studies in Nat. Mater. 12, 108 (2013) and Nat. Phys. 6, 192 (2010). We argue that the scattering matrix provides a comprehensive and effective means to illustrate these phenomena, and the inclusion of the field distribution within the waveguide does not qualitatively change our theoretical results. This is supported by our newly added calculation using finite element method approach. Specifically, in Section 2.1 of the revised manuscript, we have included a finite element method analysis of the single waveguide system, in particular, we have added new results in Fig. 1 using the finite element method. By comparing (see Fig. A) the calculation results obtained from this analysis with those derived from the scattering matrix, we have successfully confirmed the accuracy of the scattering matrix method. This addition strengthens the credibility of our research.

On the other hand, we deeply appreciate the reviewer's valuable criticism, addressing which has helped us to improve our manuscript.

**Comment 3: "The authors have made many assumptions and simplifications in this MS, which cannot be realized in experiments, or, which need numerical verifications. For example, a constant background loss of g=1 is utilized, the reason of which is not explained. The authors assumed that the coupling t between WGs is a constant of t=1, which is also not proved in numerical simulation."**

**Response 3**: We sincerely appreciate the thoughtful comment from the reviewer. The reviewer is concerned that our manuscript contains a number of assumptions and simplifications which may not be fully realizable in practical experiments. Two specific examples were mentioned: (1) The assumption of a constant background loss with $g = 1$; (2) The assumption that the coupling between waveguides is $t = 1$. Below we clarify the experimental work on which the theory is based, analyze the assumptions and simplifications used in the manuscript, and respond to the two examples proposed by the reviewer to illustrate the generality of our theoretical approach.

Our theoretical research is based on two experimental works: Nat. Mater. 12, 108 (2013) and Nat. Phys. 6, 192 (2010). They experimentally achieved refractive index modulation of a single waveguide and coupling between two waveguides respectively. The assumptions we have made are precisely based on these two works.

In our manuscript, we made three assumptions: (1) Considering that the modulation refractive index $\Delta n(z)$ is much smaller than the base refractive index $n_0$, we assume that the electric field in the waveguide only exists in two modes, $e^{ikz}$ and $e^{-ikz}$; (2) We assume that the attenuation and mode coupling between forward and backward fundamental modes mentioned in Nat. Mater. 12, 108 (2013) are both 1, which merely alters the numerical changes of the elements in the Hamiltonian matrix. We can always choose an appropriate modulation refractive index to correct it. (3) Based on Nat. Phys. 6, 192 (2010), we assume that the coupling between the two waveguides is $t$, and there is coupling only between the propagation modes in the same direction.

Regarding the first example mentioned by the reviewer (i.e., the constant background loss $g = 1$), we clarify that $g$ is not a constant background loss but a constant in the imaginary part of the modulated refractive index. It only appears in the modulation range, that is, $4n\pi/q + \pi/q \leq z \leq 4n\pi/q + 2\pi/q$, and is a parameter that can be experimentally adjusted.

Regarding the second example mentioned by the reviewer (i.e., the coupling between waveguides is $t = 1$), we clarify that $t$ is also an adjustable parameter for the experiment. In the manuscript, we choose $t = k_0/4$ to demonstrate the essential results, where $k_0$ is the wave vector in vacuum. However, we note that our results can be generalized to other parameter regimes, as one can proportionally increase or decrease the parameters $g$, $p$, $m$ and $t$ in the Hamiltonian $H_d$, which will not change the observed physical phenomena.

Based on the above, we believe the experimental setup we have proposed is feasible and the essential results can be implemented in the experiment.

**Summary: "Based on above consideration, the MS in the present form cannot be accepted. The authors could resubmit this MS after major revision."**

**our reply**: We are truly grateful for the insightful and thoughtful comments provided by the reviewer, addressing which has significantly improved our manuscript. We hope that in the light of our above clarifications on the novelty and generalizability of our results, we can convince the reviewer that our revised manuscript is suitable for publication in *Symmetry*.

## SUMMARY OF MAJOR REVISIONS

In the revised manuscript, changes are marked by blue color.

1. We rewrote the abstract and the introduction.

2. In Section 2.1, we briefly outlined the derivation process of Eqs. (5) and (6).

3. In Section 2.1, we have added statements about the approximations used in the manuscript.

4. In Section 2.1, we have added the derivation of the single waveguide scattering matrix.

5. In Section 2.1, we have added an introduction of the finite element method and Fig. 1, which presents a numerical comparison between the finite element method and the coupled mode method.

6. In Section 2.2, we have added the reasons for choosing refractive index modulation $n_1(z) = -n_2(z)$.

7. We have corrected the incorrect description of the modulated refractive index in Fig. 2 (Fig. 1 in the original manuscript).

8. We have added the discussion on the physical significance of the eigenvalue splitting of Hamiltonian.

9. In Section 2.2, we have added the discussion on the cases where the coupling coefficient $t$ between waveguides tends to 0 and $\infty$.

10. We have added Table 1 to introduce the symbols used in the manuscript.

11. In Section 3, we have added the discussion on the total energy output.

12. In Section 3.3, we have added the discussion on the restrictions on the occurrence of divergence points.

13. We have added Table 2 to summarize the key behaviors in different regions.

14. In Section 4, we have added the discussion on the feasibility of the experiment and future research.

15. We removed six references Refs. [12-14], [16], [18-19] from the original manuscript and added thirteen references Refs. [23-25], [27-31], [34], [39], [42].