Detailed Response to Reviewer 1

Open Review	(x) I would not like to sign my review report ( ) I would like to sign my review report
Quality of English Language	(x) The quality of English does not limit my understanding of the research. ( ) The English could be improved to more clearly express the research.

	Yes	Can be improved	Must be improved	Not applicable
Does the introduction provide sufficient background and include all relevant references?	( )	( )	(x)	( )
Is the research design appropriate?	( )	(x)	( )	( )
Are the methods adequately described?	( )	(x)	( )	( )
Are the results clearly presented?	( )	(x)	( )	( )
Are the conclusions supported by the results?	( )	( )	(x)	( )

Original Manuscript ID: mathematics-3271589

Original Article Title: "A Formal Fuzzy Concept-based Approach for Association Rule Discovery with Optimized Time and Storage"

Dear Reviewer,
The authors would like to thank you for your valuable comments on our submitted manuscript. We are most grateful for the time you spent providing suggestions on how to improve our paper. In this revision, we have tried our best to address the suggestions in detail below.

Kind Regards

Authors

Point-to-Point Responses to Reviewer #1:

We appreciate the time and effort that the reviewer spent on providing us with valuable improvements to our submitted manuscript.

1	**Comment**	**I suggest the author to list all abbreviations and their corresponding full names in a table at the beginning of the paper.**
	Response	Thank you for this constructive feedback. We have added the list of abbreviations table at the beginning of the paper.
Abstract		
1	**Comment**	**The abstract should point out better what are the prime objectives of this study.**

	Response	We express our gratitude to the reviewer for this note. We revised the abstract to explicitly state that the main objectives of this study, which are to enhance the extraction and visualization of frequent fuzzy closed item sets and to optimize time and storage requirements.
2	**Comment**	**Authors should depict a brief explanation about the necessity of carrying out such study.**
	Response	We amended the abstract to explain the need for this study by highlighting the computational challenges of traditional fuzzy approaches, such as high time and storage demands, and their impracticality for handling heterogeneous data types.
3	**Comment**	**The conclusion should be explained**
	Response	The conclusion has been expanded to include the findings, emphasizing the superior performance of the fuzzy iceberg lattice in comparison to traditional methods in terms of time and space efficiency.
1. Introduction		
1	**Comment**	**Please specify in the introduction, based on the relevant literature, the research gap. Here author must build research gap following the previous studies. The manuscript does not answer the following concerns: Why is it timeliness to explore such a study? What makes this study different from the previously published studies? Are there any similarly findings in line with the previously published studies? Are the findings different from prior academic studies that were conducted elsewhere, if any? For example, what it requires, what are the new technologies, some recent issue highlights the importance.**
	Response	We sincerely appreciate this comment. We have addressed the research gap in the revised version by emphasizing the timeliness of this study. Specifically, we highlight the growing need for efficient ARM techniques in the context of handling large-scale data across diverse industries, making the research relevant and timely, on the revised version, page 2, 2nd, 3rd, and 4th paragraphs. We have also clarified how our study differs from previous work by introducing the concept of a fuzzy iceberg lattice that selectively retains only the essential portion of the concept lattice. This method optimizes the ARM process by focusing on frequent closed itemsets above a predefined support threshold, contrasting with traditional methods that construct complete concept lattices, on page 2, 4th and 5th paragraphs

		Additionally, the introduction discusses how the proposed method addresses challenges seen in prior studies, such as computational inefficiency and memory consumption when handling large datasets. We explicitly compare our method to existing studies that have applied FFCA in ARM and show that our approach is more computationally efficient and scalable.
2	**Comment**	**There are no research questions clearly delineated and no discussion how author is going to address these research questions. I am sorry but it is quite hard to follow. Also, there is no discussion about how authors are going to address these research questions. This is required, because it helps the reader to understand your research better.**
	Response	We sincerely appreciate this comment We have explicitly added a section to the revised introduction where we clearly delineate the research questions as follows: 1. How does incorporating a user-centric approach, which considers subjective beliefs about data, impact the interpretability and effectiveness of extracted rules? 2. How to optimize the process of extracting association rules without losing critical information? 3. What improvements can be achieved in computational efficiency and rule compactness when using the proposed fuzzy iceberg lattice over conventional methods? We have also outlined how these research questions will be addressed through our proposed methodology. The steps of the approach— (1) mapping the dataset to a fuzzy representation using user-centric linguistic labels, enabling an interpretable transformation to a fuzzy variant without information loss; (2) constructing a fuzzy iceberg lattice focused on frequent closed itemsets, which leverages crisp intents and their corresponding fuzzy supports to filter out infrequent concepts early, thereby optimizing the extraction of association rules; and (3) generating concise fuzzy implication and association rules through fuzzifying a rule-based extraction algorithm that ensures no redundant rules are included. This approach ensures that the research questions are not only defined but also addressed comprehensively throughout the paper. ***Additional Clarifications:*** We have now made the structure of the study and the methodology easier to follow. By clearly outlining the contributions and providing a more logical flow of ideas in the revised introduction, we aim to make the paper more understandable and aligned with the expectations of the reviewer. Additionally, the distinction between our approach and those in previous

		studies is now more explicitly discussed, answering concerns about what makes this study different and timely. Thank you for the valuable feedback.
2. Classical and Fuzzy Formal Concept Analysis		
1	**Comment**	**Please provide a detailed explanation of the formula used to calculate the data in Table 4 and obtain the data in Table 5, line 145-154.**
	Response	We thank the reviewer for his valuable comment to enhance the clearance of our manuscript. We have revised this part and added the required explanation on page 6.
2	**Comment**	**Please provide a detailed explanation of the concepts in Definition 5, such as fuzzy context, fuzzy concept, and critic intent, based on Example 1**
	Response	We sincerely appreciate this insightful comment. We have clarified the definition of the fuzzy context using Example 1 in the 4th paragraph on page 6. Additionally, the concepts of fuzzy concept and intent introduced in Definition 5 are elaborated upon through Example 1 in the last paragraph of page 7.
3	**Comment**	**How was the data in Figure 3 obtained? Please provide detailed calculation steps**
	Response	Thank you for your valuable comment. We used the attribute-based algorithm [20] to construct the fuzzy lattice in figure 3. Therefore, we provided details for describing how this concept lattice is generated in 4th paragraph on page 7.
3. Literature Review		
1	**Comment**	**What's the difference between crisp-fuzzy FFCA and fuzzy-crisp FFCA? It would be best if the author could use a figure to show the difference between the two.**
	Response	Thank you for your insightful comment. The distinction between crisp-fuzzy FFCA and fuzzy-crisp FFCA lies in the handling of the extent and intent within the fuzzy concept: • Crisp-fuzzy FFCA: In this approach, the extent (the set of objects) is crisp (precise), meaning each object either fully belongs to the concept or does not. However, the intent (the set of attributes) is fuzzy, meaning

		the degree to which an attribute belongs to the concept varies continuously for each object. • Fuzzy-crisp FFCA: Here, the extent is fuzzy, meaning the objects have a gradual degree of membership in the concept. The intent, on the other hand, is crisp, meaning the attributes are clearly defined without any intermediate degrees of membership. We have clarified these differences in 2nd paragraph on page 5. Besides, we have included figure 1 that visually represents the two approaches. Again, thank you for suggesting this improvement.
5. Results and Discussion		
1	**Comment**	**1. Through method comparison, it can be concluded that the method proposed by the author has high computational efficiency?**
	Response	Thank you for your comment. The comparative analysis of methods indeed suggests that the proposed approach demonstrates high computational efficiency. This is evident from the significant reduction in time and memory required for generating fuzzy association rules using the proposed fuzzy-crisp iceberg lattice, compared to the time needed to build the entire set of fuzzy concepts using fuzzy concept lattice.
2	**Comment**	**It is necessary for the author to provide the calculation process involved in the entire text for reproduction and verification.**
	Response	The primary purpose of the algorithm is to present the underlying conceptual approach to fuzzy association rule discovery, focusing on the theoretical foundation rather than the specific implementation steps. Therefore, while the algorithm provided outlines the process conceptually, it is designed to be adaptable across different execution methods. However, to support reproduction and verification, we would be happy to provide additional details on the calculation process, highlighting the core steps and assumptions involved, ensuring clarity without prescribing a specific coding approach. Additionally, we work on and plan to publish a programming package that implements SOTA methods as well as our proposed approach.
6. Conclusions		

1	**Comment**	**It is necessary for the author to further explain the innovative points and contributions, as well as the shortcomings of the research in the conclusion**.
	Response	Thank you for your valuable feedback. We have revised and amended the conclusion, we have elaborated on the innovative aspects of our approach, specifically the introduction of the fuzzy-crisp iceberg lattice structure and the fuzzy min-max basis algorithm. We clarified how these contributions advance association rule mining (ARM) by enabling efficient handling of heterogeneous data and reducing redundancy in rule generation. Additionally, we have acknowledged the primary shortcoming of the current approach: its lack of support for incremental updates, which could limit its effectiveness in dynamic data environments. Future research directions to address this limitation have also been discussed. We believe these revisions provide a clearer and more comprehensive overview of our contributions and limitations.