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Referee's report

Title: Infinite Series Concerning Tails of Riemann Zeta Values

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Work submitted to *Axioms*.

According to the abstract the article deals to establish twelve general summation theorems that express these series as coefficients of the bivariate beta function $Beta(u, v)$ using integral representations of zeta tails together with the telescopic approach.

The results presented are novel. However, the process of understanding the proofs and even some steps are not clear so they must be clarify before this work is considered for publication.

I will strongly suggest some modifications to improve/correct some results in the paper:

1. In the keywords (it is not necessary to use bold font), you should split Gamma function, and Beta function. Amend this.
2. Replace MSC by "2020 Math. Subject Classification".
3. You must present proof of the switch between the sum and the integral of equation (5), and add the code of the © Wolfram Mathematica file as an appendix of this work.
4. Page 3, line 2 (P3L2): present a reference, or give complete proof, of the equality

$$\sigma(m, n) = [x^m] \binom{x+n}{n}$$

5. Page 3, line 4 (P3L4): present a reference, or give complete proof, of the presented recurrence relation for the $\sigma(m, n)$ function.
6. P3L5: write [19, §3.3] using the citation system properly.
7. Give a more detailed proof of Lemma 1, adding the proper constraints of the variable y , probably it is enough $|y| < 1$. What about x in the proof? Explain with more detail section (a).

8. P3L19: Remove the symbol $\times$, it is a standard product.
9. You should prove that the integral at the bottom of page 3 is convergent and present similar arguments for the rest of the improper integrals in this work.
10. Page 4, lines 2-3: present a reference, or give complete proof, of the equalities.
11. P4L6: write "Then, by extracting the of $u^m v^n$, we have". You should be more explicit when extracting the coefficients of some expressions. Amend this along this work.
12. State properly Theorem 2: "For any $m, n \in \mathbb{N}_0$, the following identity holds:". Amend this along the manuscript.
13. P4L16: Write "Some particular cases related to this theorem are the following ones". Amend this along this work.
14. Present a complete proof of Corollary 3 or give enough information to obtain such values.
15. Give a complete sketch or proof of the exponential generating function, explaining how to interchange sums and integrals in such a case.
16. State properly Theorem 4.
17. Present a complete proof of Corollary 5 or give enough information to obtain such values.
18. P6L11 and P6L24: Remove the symbol $\times$, it is a standard product.
19. State properly Theorem 6.
20. Present a complete proof of Corollary 7 or give enough information to obtain such values.
21. You should apply similar replacements along the rest of the manuscript.
22. Give a complete proof or present references about the results presented in section 3, on page 8.
23. State lemma 10 properly, add the constraints related to the variables of the result and present a complete proof of it.

At this point, it is clear the author should be more explicit providing complete proof of all the novel results and identities he is presenting, stating the results correctly, and giving the sketch or a clear idea about how to obtain such identities. The reader does not need all such amount of particular cases. Nevertheless, it is fine to have them if the extension is not an issue for the publication, and it would be great if the author provides a ©Wolfram Mathematica file or his code about obtaining all these values. Finally, the references should be arranged alphabetically.

With all of this, the results are exciting, they seem novel. Unfortunately, the manuscript needs a major revision before I can be willing to accept this manuscript for publication in **AXIOMS**.