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Dear Editor,

I have read the article entitled with *"Nonlinear charged black hole solution in Rastall gravity"* in detail.

Author has demonstrated that the Rastall parameter has no effect on the spherically symmetric black hole solution of the charged (linear case) field equation of the Rastall gravitational theory, which is consistent with the results reported in the literature. Moreover, author derived a novel spherically symmetric black hole solution that involves the Rastall parameter when we use the field equation of Rastall's theory with a certain type of NED (nonlinear electrodynamics) source. The trace portion of the NED source, which unlike the linear charged field equation has a non-vanishing value, is the primary cause of the development of this parameter. It is also demonstrated that the Anti-de Sitter Reissner-Nordström (AdSRN) spacetime in the new black hole solution is simply cosmic constant-absorbent AdSRN spacetime. In the GR limit, or when Rastall's parameter vanishes, this solution simply reduces to the RN solution. The stability is investigated for utilizing the deviation of geodesic equations to obtain the stability criterion to learn more about this BH. Additionally, Author explored BH's thermodynamical characteristics to determine its stability in contrast to the linear charged instance, which experiences a second-order phase shift. The first law of thermodynamics is also tested for the obtained black hole.

Rastall gravity, originally developed in 1972, is currently undergoing a significant surge in popularity. Rastall gravity purports to be a modified theory of gravity, with a non-conserved stress–energy tensor, and an unusual non-minimal coupling between matter and geometry.

I have found the study physically correct and valuable. I have checked that the resulting equations and their solutions are (in general) correct.

However, as can be seen in the attached part of the report, there are a number of typographical errors (which I am sure will find many more if examined in detail). They must be fixed.

I see that applications (in the literature) related to Rastall gravity are not mentioned enough. This reduces the importance of the subject. My recommendation is to include the articles seen below:

1. Shadow cast of non-commutative black holes in Rastall gravity

Modern Physics Letters A 2050163 2020

2. Quasinormal Modes of Black Holes with Non-Linear-Electrodynamic sources in Rastall Gravity

Physics of the Dark Universe, 100860 (2021)

3. Dirac quasinormal modes of power-Maxwell charged black holes in Rastall gravity

2005.00674v1

4. Quasinormal Modes of Non-Linearly Charged Black Holes surrounded by a Cloud of Strings in Rastall Gravity

arXiv: 2111.00854v1

5. Quasinormal mode and spectroscopy of a Schwarzschild black hole surrounded by a cloud of strings in Rastall gravity

Phys. Rev. D 101, 104023 (2020)

6. Black hole solutions and Euler equation in Rastall and generalized Rastall theories of gravity

Mod. Phys. Lett. A, 34 1950304 (2019)

7. Rastall cosmology

Int. J. Mod. Phys. Conf. Ser., 18, 67-76 (2012)

8. GUP Modified Hawking Radiation in Bumblebee Gravity

Nuclear Physics B Volume 946, September 2019, 114703

9. Gravitational Collapse in Rastall Gravity

Physics Letters B 793 (2019) 276

10. Neutron Stars in Rastall Gravity

Phys. Rev. D, 92 (2015)

Some statements about the future of this study should also be included in the Conclusion section.

Finally, according to my point of view, this article needs “Moderate Revision”.

Nonlinear charged black hole solution in Rastall gravity

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(Dated: August 10, 2022)

We show that the spherically symmetric BH solution of a charged (linear case) field equation of Rastall gravitational theory does not affect by the Rastall parameter and this is consistent with the results presented in the literature. However, when we apply the field equation of Rastall's theory with a special form of nonlinear electrodynamics (NED) source we derive a novel spherically symmetric BH solution that involves the Rastall parameter. The main source of the appearance of this parameter is the trace part of the NED source which has a non-vanishing value, unlike the linear charged field equation. We show that the new BH solution is just Anti-de-Sitter Reissner-Nordström spacetime in which the Rastall parameter is absorbed into the cosmological constant. This solution coincides with Reissner-Nordström solution in the GR limit, i.e. when Rastall's parameter vanishing. To gain more about this BH, we study the stability using the deviation of geodesic equations to derive the stability condition. Moreover, we explain the thermodynamical properties of this BH and investigate that it is stable, unlike the linear charged case that has second-order phase transition. Finally, we prove the validity of the first law of thermodynamics.

black hole (BH)

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I. INTRODUCTION

Since the construction of Einstein's general relativity (GR), the coupling between a scalar field and the gravitational action in a geometric frame has been intensively studied. A formulation of scalar theory was done in [1], and later Jordan-Brans-Dicke constructed a gravitational theory as an expansion of GR to investigate the variable of gravitational coupling [2–4]. Afterward, a general combination between a scalar field and its derivative, which yields second-order differential equations, is known as Horndeski theory [5] that gained much attention. Recently, many modifications of Einstein GR are established. Among these theories is the one known as $f(R)$ gravitational theory, which is considered a natural generalization of Einstein Hilbert action [6]. This theory could be rewritten as a GR and scalar field [7, 8]. The above is a very brief summary related to the scalar fields in the frame of gravitational context however, there is a huge literature on this subject.

The above discussions show one way of modification of GR, however, there is another possibility that has been used to generalize the kinetic term of the scalar field that is minimally coupled to the Einstein-Hilbert action. This possibility is called the k-essence theories [9]. This theory is used as an option to the usual inflationary models that used a self-interacting scalar field [9–12]. Recently, vacuum static spherically symmetric solutions have been derived for the k-essence theories [12]. Some novel patterns have been derived that involve a study of the event horizon. Nevertheless, interpolating such solutions as black holes was difficult because it is impossible to define a distant region from the horizon. Using the no-go theorem it has been affirmed that solutions with a regular horizon can exist but only of the type of cold black hole [13, 14].

Another generalization of GR is to abound the restriction of the conservation law that encoded in the zero divergence of the energy-momentum tensor. Among the theories that follow this direction is the one given by Rastall (1972) which is known as Rastall's theory [15]. In the frame of Rastall theory the covariant divergence of stress-energy momentum tensor is proportional to the covariant divergence of the curvature scalar, i.e. $T^{\alpha}{}_{\beta;\alpha} \propto R_{;\beta}$. Thus any solution that has a zero or constant Ricci scalar Rastall theory will be identical to Einstein GR. Explaining the behavior of the new source of Rastall theory is not an easy task. We can consider, phenomenologically, this new source as an appearance of quantum effects in the classical frame [16]. It is interesting to mention that the topic of non-conservation of $T^{\alpha\beta}$ is a feature exists in diffusion models [17–21]. Also, the non-conservation of the energy-momentum tensor and its link to modified gravitational theories has been analyzed in [22, 23]. The variational principle in the frame of Riemannian geometry is not held due to the non-conservation of $T^{\alpha\beta}$. Nevertheless, some features like Rastall's theory can also be discovered in the frame of Weyl geometry [24]. Moreover, external fields in the Lagrangian could give essentially the same behavior as Rastall's theory (for discussion of the external field see for example [25]).

In the frame of cosmology, Rastall's theory could degenerate with the Λ cosmological dark matter, ΛCDM, at the background and at first order levels which means that a viable model can be constructed in the frame of this theory. However, a few application in the domain of astrophysics has been done [26]. Recently, various BH solutions, and in particular, BH solutions of the Rastall field equationshave been investigated in many scientific research papers. Among these is the one include charged static spherically symmetric BH solutions [27, 28], Gaussian BH solutions [29, 30], rotating BH solutions [31, 32], Abelian-Higgs strings [33], G$\ddot{o}$del-type BH solutions [34], black branes [35], worm holes [36], neutron stars [37], BH solutions surrounded by fluid, electromagnetic field [38] or quintessence fluid [39], BH thermodynamics [40], higher-dimensional BH solutions [39], among other theoretical efforts [41–44]. It is the aim of the present study to show the effect of the Rastall parameter in the domain of spherically symmetric spacetime using a special form of NED coupled with Einstein's GR.

This paper has the following structure: In the next section, we give a summary of Rastall's theory. In Subsection II A we give the NED field equations of Rastall's gravity then we apply them to a spherically symmetric spacetime with two unequal metric potentials and derive the NED differential equations. We solve this system and derive a new BH solution that involves Rastall's parameter. In Subsection II B, we extract the physical properties of the BH solution and show that the metric potentials asymptote as Anti-de-Sitter (A)dS Reissner–Nordström. In spite that we applied the NED field equations without cosmological constant, we get (A)dS Reissner–Nordström. This means that the Rastall parameter acts as a cosmological constant in this special form of NED theory. This result is consistent with the study given by Visser [45]. It is important to stress that this solution in the GR limit, i.e., when the Rastall parameter equals zero, coincides with Reissner-Nordström solution. In Subsection II C we derive the stability of geodesic motion using the geodesic deviations. In Section III, we study some thermodynamical quantities and in Subsection III A, we show that our BH satisfies the first law of thermodynamics. In Section IV we discuss the output results of this study.

II. SPHERICALLY SYMMETRIC BH SOLUTION

According to Rastall's assumptions [15, 46], for a spacetime with Ricci scalar R filled by an energy-momentum $T_{\mu\nu}$, we have

$$T^{\alpha\beta}{}_{;\alpha} = \epsilon \mathcal{R}_{;}{}^{\beta}\,, \tag{1}$$

where ϵ is the Rastall parameter, which responsible for the deviation from the standard GR conservation law. Equation (1) returns to Einstein's GR when Ricci scalar is vanishing or has a constant value.

Using the above data, we can write Rastall field equations in the form [15, 46]:

$$\mathcal{R}_{\alpha\beta} - \left[\frac{1}{2} + \lambda\right] g_{\alpha\beta}\mathcal{R} = \chi T_{\alpha\beta}\,, \tag{2}$$

where $\lambda = \chi\epsilon$ and χ is the Newtonian gravitational constant and the units are used so that the speed of light $c = 1$. Here $\mathcal{R}_{\alpha\beta}$ is the Ricci tensor, $\mathcal{R}$ is the Ricci scalar, $g_{\alpha\beta}$ is the metric tensor, and $T_{\alpha\beta}$ is the energy-momentum tensor describing the material content. The constant ϵ is the Rastall parameter that is responsible for the deviation from GR and when $(\epsilon = 0)$ we get GR theory.

The modification in the spacetime geometry given by the L.H.S. of Eq. (2) links to two modifications of different material content of the right hand side of Eq. (2):

(i) Firstly, Eq. (2) is mathematically equivalent to adding new material of the actual material sources to the right-hand side of the standard GR field equations, which can be seen as an effective source accompanying the actual material sources considered in the model. For this reason, we can rewrite Eq. (2) in a mathematical equivalent form as [15, 46]:

$$\mathcal{R}_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}\mathcal{R} = \chi T_{1\alpha\beta}\,, \qquad \text{where} \qquad T_{1\alpha\beta} = T_{\alpha\beta} - \frac{\chi\,\epsilon}{1+4\chi\,\epsilon}g_{\alpha\beta}T\,. \tag{3}$$

The term $-\frac{\epsilon}{1+4\epsilon}g_{\alpha\beta}T$ is the energy-momentum tensor represents the effective source that arises from the actual material and T is the trace of $T_{\alpha\beta}$, i.e., $T = g_{\alpha\beta}T^{\alpha\beta} = -(1+4\epsilon)R$.

Now we are going to rewrite Eq. (3) in the form:[1]

$$\mathcal{R}_{\alpha\beta} - g_{\alpha\beta}\mathcal{R}\left[\frac{1}{2} + \epsilon\right] \equiv \mathcal{R}_{\alpha\beta} + g_{\alpha\beta}T\left[\frac{1+2\epsilon}{2(1+4\epsilon)}\right] = T_{\alpha\beta}\,. \tag{4}$$

In this study, we are going to use Eq. (4) but we assume the energy-momentum tensor $T_{\alpha\beta}$ to be combine with electromagnetic field and takes the following form:

combined

$$T_{\alpha\beta} = E_{\alpha\beta}, \qquad \text{where} \qquad E_{\alpha\beta} = F^{\mu}{}_{\alpha}F_{\mu\beta} - \frac{1}{4}g_{\alpha\beta}F\,, \tag{5}$$

with $F_{\mu\beta}$ being the antisymmetric Faraday tensor and $F = F^{\mu\nu}F_{\mu\nu} = d\xi$ and $\xi = \xi_\alpha dx^\alpha$ is the electromagnetic gauge potential Maxwell field [47]. The tensor $F_{\mu\beta}$ satisfies the vacuum Maxwell equations

$$F^{\alpha\beta}{}_{;\alpha} = 0\,, \qquad F_{\alpha\beta;\sigma} + F_{\beta\sigma;\alpha} + F_{\sigma\alpha;\beta} = 0\,. \tag{6}$$

ii)-Secondly, this modification implies a violation of the local conservation of the tensor $T_{1\alpha\beta}$ of an actual material source because its divergence is not necessarily to be vanishing.

It is important to stress that Eq. (4) with the energy-momentum tensor given by Eq. (6) has a contradiction since the LHS of Eq. (4) has a non-vanishing of covariant derivative, $\left\{\mathcal{R}^{\alpha\beta} - g^{\alpha\beta}\mathcal{R}\left[\frac{1}{2} + \frac{\epsilon}{1+4\epsilon}\right]\right\}_{;\beta} \neq 0$ while the RHS has a vanishing value, $T^{\alpha\beta}{}_{;\beta} = 0$. Thus, the only way to overcome this issue is the fact that the solution of these field equations must have a zero Ricci scalar[2] which ensures the well known results in the literature that the Rastall parameter has no effect in linear Maxwell field.

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[1] In this study we assume the relativistic units, i.e., $\chi = \frac{8\pi G}{c^4} = 1$.

[2] In the frame of Rastall theory Reissner-Nordström is a solution since its Ricci scalar has a vanishing value.

A. Nonlinear charged spherically symmetric BH solution in Rastall's theory

In this subsection, we are going to present a special form of a NED theory coupled with GR. For this aim, we are going to take into account a dual representation, i.e., imposing the auxiliary field $\mathcal{S}_{\alpha\beta}$, which is convenient to couple with GR [48, 49]. Especially, we involve the Legendre transformation

$$\mathbb{H} = 2FL_F - L, \tag{7}$$

where $\mathbb{H}$ is an arbitrary function, $L_F \equiv \frac{\partial L}{\partial F}$ and $L(F) = 4F$. Assuming

$$\mathcal{S}_{\mu\nu} = L_F F_{\mu\nu}\,, \qquad \mathcal{S} = \frac{1}{4}\mathcal{S}_{\alpha\beta}\mathcal{S}^{\alpha\beta} = L_F^2 F\,, \qquad \text{with} \qquad F_{\mu\nu} = \mathbb{H}_{\mathcal{S}}\mathcal{S}_{\mu\nu}\,, \tag{8}$$

where $\mathbb{H}_{\mathcal{S}} = \frac{\partial \mathbb{H}}{\partial \mathcal{S}}$. The field equation of nonlinear electrodynamics yields the form [48]:

$$\partial_\nu\left(\sqrt{-g}\mathcal{S}^{\mu\nu}\right) = 0, \tag{9}$$

where the energy-momentum tensor of the NED is defined as:

$$T^{\nu}{}_{\mu}^{NED} \equiv 2(\mathbb{H}_S S_{\mu\alpha}S^{\nu\alpha} - \delta^\nu_\mu[2S\mathbb{H}_S - \mathbb{H}]). \tag{10}$$

We mention that in general Eq. (10) has a non-vanishing trace[3]

$$T^{NED} = 8(\mathbb{H} - \mathbb{H}_{\mathcal{S}}\mathcal{S}) \neq 0\,, \tag{11}$$

and has a vanishing value in the linear theory, i.e., when $\mathbb{H} = -2F$ and $\mathcal{S} = F$. Finally, the electric and magnetic fields in the NED case take the form [48, 49]:

$$\begin{aligned} E &= \int F_{tr}dr = \int \mathbb{H}_S S_{tr}dr\,, & B_r &= \int F_{r\phi}d\phi = \int \mathbb{H}_S S_{r\phi}d\phi, \\ B_\theta &= \int F_{\theta r}dr = \int \mathbb{H}_S S_{\theta r}dr, & B_\phi &= \int F_{\phi r}dr = \int \mathbb{H}_S S_{\phi r}dr\,, \end{aligned} \tag{12}$$

where E and B are the components of the electric and magnetic fields respectively. Now we are going to use the field equation (4) with the energy-momentum tensor $T_{\alpha\beta}$, that is combined with the NED, and get:

$$T_{\mu\nu}^{NED} \equiv \mathbb{E}_{\mu\nu}, \qquad \text{where} \qquad \mathbb{E}_\mu{}^\nu = 2(\mathbb{H}_S S_{\mu\alpha}S^{\nu\alpha} - \delta^\nu_\mu[2S\mathbb{H}_S - \mathbb{H}])\,. \tag{13}$$

Now, let us assume that the spherically symmetric spacetime has the form:

$$ds^2 = -\mu(r)dt^2 + \frac{dr^2}{\nu(r)} + r^2(d\theta^2 + d\phi^2)\,, \tag{14}$$

where $\mu(r)$ and $\nu(r)$ are unknown functions of the radial coordinate r. For the spacetime (14), the symmetric affine connection takes the form:

$$\begin{aligned} &\Gamma_{tt}{}^r = \frac{1}{2}\nu\mu'\,, && \Gamma_{tr}{}^t = \frac{\mu'}{2\mu}\,, && \Gamma_{rr}{}^r = \frac{\nu'}{2\nu}\,, && \Gamma_{r\theta}{}^\theta = \Gamma_{r\phi}{}^\phi = \frac{1}{r}\,, \\ &\Gamma_{\theta\theta}{}^r = -r\nu\,, && \Gamma_{\theta\phi}{}^\phi = \cot\theta\,, && \Gamma_{\phi\phi}{}^r = -r\nu\sin^2\theta\,, && \Gamma_{\phi\phi}{}^\theta = -\sin\theta\,\cos\theta\,. \end{aligned} \tag{15}$$

The Ricci scalar of the spacetime (14) has the form:

$$\mathrm{R(r)} = \frac{r^2\nu\mu'^2 - r^2\mu\mu'\nu' - 2r^2\mu\nu\mu'' - 4r\mu[\nu\mu' - \mu\nu'] + 4\mu^2(1-\nu)}{2r^2\mu^2}\,. \tag{16}$$

Here, $\mu \equiv \mu(r)$, $\nu \equiv \nu(r)$, $\mu' \equiv \frac{d\mu}{dr}$, $\mu'' \equiv \frac{d^2\mu}{dr^2}$ and $\nu' \equiv \frac{d\nu}{dr}$.

[3] The non-vanishing of the trace is an important property in the frame of Rastall's theory so that the effect of Rastall parameter maybe appear unlike Maxwell field theory.

Using Eq. (14) in Eq. (4), where the energy-momentum tensor is given by Eq. (13), we get:

The t t-component of Rastall field equation is:

$$\frac{1}{2r^2\mu^2[2\mu\nu\xi''+\xi'(\mu\nu'-\nu\mu')]}\Big\{2\mu\nu\xi''\left\{2\mu\nu\epsilon r^2\mu''-r^2\nu\epsilon\mu'^2+r\mu\mu'\epsilon[4\nu+r\nu']+2\mu^2[(1+2\epsilon)[r\nu'+\nu]+1+r^2\mathbb{H}+2\epsilon]\right\}$$
$$+\xi'\Big(2r^2\mu\nu\mu''\epsilon[\mu\nu'-\nu\mu']+r^2\nu^2\epsilon\mu'^3-2r\mu\nu\mu'^2[r\nu'+2\nu]\epsilon+\mu^2\mu'\left[r^2\epsilon\nu'^2-2r\nu\nu'-2\nu(\{1+2\epsilon\}(\nu-1)-r^2\mathbb{H})\right]$$
$$+\mu^3\left\{2\nu'[(1+2\epsilon)\{r\nu'+\nu-1\}-r^2\mathbb{H}]+r^2\nu\mathbb{H}'\right\}\Big)\Big\}=0\,,$$

The r r-component of Rastall field equation is:

$$\frac{1}{2r^2\mu^2[2\mu\nu\xi''+\xi'(\mu\nu'-\nu\mu')]}\Big\{2\mu\nu\xi''\left\{2\mu\nu\epsilon r^2\mu''-r^2\nu\epsilon\mu'^2+r\mu\mu'[2(1+2\epsilon)\nu+\epsilon r\nu']+2\mu^2[(1+2\epsilon)[\nu-1]+2\epsilon r\nu'-r^2\mathbb{H}]\right\}$$
$$+\xi'\Big(2r^2\mu\nu\mu''\epsilon[\mu\nu'-\nu\mu']+r^2\nu^2\epsilon\mu'^3-2r\mu\nu\mu'^2[r\nu'\epsilon+(1+2\epsilon)\nu]+\mu^2\mu'\left[r^2\epsilon\nu'^2+2r\nu\nu'-2\nu(\{1+2\epsilon\}(\nu-1)-r^2\mathbb{H})\right]$$
$$+2\mu^3\left\{2\epsilon r\nu'^2+\nu'[(1+2\epsilon)\{\nu-1\}-r^2\mathbb{H}]+r^2\nu\mathbb{H}'\right\}\Big)\Big\}=0\,,$$

The $\theta\,\theta=\phi\,\phi$-component of Rastall field equation is:

$$\frac{1}{4r^2\mu^2[2\mu\nu\xi''+\xi'(\mu\nu'-\nu\mu')]}\Big\{2\mu\nu\xi''\Big\{2\mu\nu(1+2\epsilon)r^2\mu''-r^2\nu(1+2\epsilon)\mu'^2+r\mu\mu'[2(1+4\epsilon)\nu+(1+2\epsilon)r\nu']+2\mu^2[4\epsilon[\nu-1]$$
$$+(1+4\epsilon)r\nu'-2r^2\mathbb{H}]\Big\}+\xi'\Big(2r^2\mu\nu\mu''(1+2\epsilon)[\mu\nu'-\nu\mu']+r^2\nu^2(1+2\epsilon)\mu'^3-2r\mu\nu\mu'^2[r\nu'(1+2\epsilon)+(1+4\epsilon)\nu]$$
$$+\mu^2\mu'\left[r^2(1+2\epsilon)\nu'^2-4\nu(2\epsilon(\nu-1)-r^2\mathbb{H})\right]+2\mu^3\left\{(1+4\epsilon)r\nu'^2+2\nu'[2\epsilon\{\nu-1\}-r^2\mathbb{H}]+4r^2\nu\mathbb{H}'\right\}\Big)\Big\}=0\,, \tag{17}$$

where $\mathbb{H}$ is an arbitrary function and ξ is the field of electric charge. Equations (17) reduce to the linear charged Einstein's field equations when $\epsilon=0$ and $\mathbb{H}=F$ [50]. The exact solution of Eq. (17) for the electric field takes the form:[4]:

$$\mu(r)=\frac{c_2(c_3r^4+(1+4\epsilon)[12r^2+12rc_4-3c_5])}{r^2}\,,\qquad \nu(r)=\frac{c_3r^4+(1+4\epsilon)[12r^2+12rc_4-3c_5]}{12r^2(1+4\epsilon)}\,,$$
$$\xi(r)=\frac{c_1}{r}\,,\qquad \mathbb{H}=c_3+\frac{c_5}{r^4}\,. \tag{18}$$

Equation (18) shows that the Rastall parameter has an effect in the NED case. When $\mathbb{H}=F\equiv\frac{c_5}{r^4}$ we return to the linear charged case [51]. We stress the fact that if we repeat the same above calculations taking into account the electric and magnetic fields, given by Eq. (12), we can easily verify the same conclusion of the above case, i.e., Rastall's parameter has an effect and its behavior will be similar to the form given by Eq. (18).

B. The physical properties of the BH solutions (18)

Now we are going to explain the physics of the BH solution (18). For such purpose, we rewrite the components of the metric potential of the BH (18) as:

$$\mu(r)=\nu(r)=r^2\Lambda_{eff}+1-\frac{2M}{r}+\frac{q^2}{r^2}\,,\qquad \xi=-\frac{q}{r}\,,\qquad \mathbb{H}=12\Lambda_{eff.}(1+4\epsilon)-\frac{4q^2}{r^4}\,. \tag{19}$$

where we have put

$$c_1=-q\,,\quad c_2=\frac{1}{12(1+4\epsilon)}\,,\quad \Lambda_{eff.}=c_3c_2\,,\quad c_4=-2M\,,\qquad \text{and}\qquad -4q^2=c_5\,,\quad c_5=-4\sqrt{c_1}\,. \tag{20}$$

[4] Solution (18) has been checked using Maple software 19.

Equation (20) shows that we have got an effective cosmological constant in the solution of NED charged case while their field equations have no cosmological constant. This means that the Rastall parameter acts as an effective cosmological constant in the NED charged case with the fact that Rastall parameter $\epsilon \neq -\frac{1}{4}$. From Eq. (19) and Eq. (14) we get:

$$ds^2 = -\left\{r^2\Lambda_{eff} + 1 - \frac{2M}{r} + \frac{q^2}{r^2}\right\}dt^2 + \frac{dr^2}{r^2\Lambda_{eff} + 1 - \frac{2M}{r} + \frac{q^2}{r^2}} + d\Omega^2\,, \tag{21}$$

where $d\Omega^2 = r^2(d\theta^2 + \sin^2 d\phi^2)$ is a 2-dimensional unit sphere.

Eq. (21) shows that solution (18) asymptotes as (A)dS and does not equal to Reissner–Nordström spacetime due to Rastall parameter. Equation (21) investigates clearly that Rastall parameter acts as a cosmological constant. Eq. (21) coincides with GR when $\mathbb{H} = F$ which means $c_3 = 0$ and this gives Rissner-Nordström BH solution because $\Lambda_{eff} = 0$.

From Eq. (19) and Eq. (16) we get:

$$\mathrm{R}(r) = -12\Lambda_{\mathrm{eff}}\,. \tag{22}$$

Equation (22), shows in a clear way that the Rastall parameter acts as a cosmological constant and the conservation law of both sides of Eq. (2) are satisfied.

Using Eq. (19) we get the invariants as:

$$\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} = -24\Lambda_{\mathrm{eff}} + \frac{48m^2}{r^6} - \frac{96mq^2}{r^7} + \frac{56q^4}{r^8}\,, \qquad \mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} = 36\Lambda_{\mathrm{eff}} + \frac{4q^4}{r^8}, \qquad \mathcal{R} = -12\Lambda_{\mathrm{eff}}\,. \tag{23}$$

Here $(\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}, \mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}, \mathcal{R})$ are the Kretschmann scalar, the Ricci tensor square, the Ricci scalar, respectively. The Kretschmann scalar, the Ricci tensor square have a true singularity when $r = 0$. All the above invariants are identical with the invariant of (A)dS Reissner–Nordström BH solution of GR. The discussion of the invariant of (A)dS Reissner–Nordström can be applied on the invariant given by Eq. (23) with the exclusion of the value $\epsilon = -\frac{1}{4}$.

Before we closing this subsection we are going to calculate the trace of the NED given by Eq(11) using solution (18) and get:

$$T^{NED} = c_3 \neq 0\,. \tag{24}$$

Equation (24) shows in a clear way that if $c_3 = 0$ we will get a vanishing trace and in that case Rastall's parameter will has no effect which support the above discussion.

C. Stability of geodesic motion of BH (19)

The equations of geodesic are given by [52]:

$$\frac{d^2x^\gamma}{d\varepsilon^2} + \left\{ \begin{array}{c} \gamma \\ \beta\rho \end{array} \right\} \frac{dx^\beta}{d\varepsilon}\frac{dx^\rho}{d\varepsilon} = 0\,, \tag{25}$$

where ε is a canonical parameter. Moreover, the equations of geodesic deviation are given as [53, 54]:

$$\frac{d^2\varrho^\sigma}{d\varepsilon^2} + 2\left\{ \begin{array}{c} \sigma \\ \mu\nu \end{array} \right\} \frac{dx^\mu}{d\varepsilon}\frac{d\varrho^\nu}{d\varepsilon} + \left\{ \begin{array}{c} \sigma \\ \mu\nu \end{array} \right\}_{,\rho} \frac{dx^\mu}{d\varepsilon}\frac{dx^\nu}{d\varepsilon}\varrho^\rho = 0\,, \tag{26}$$

where ϱ^α is the deviation of the 4-vector.

Following the procedure done in [55, 56] one can get the stability condition as:

$$\frac{3\mu\nu\mu' - \sigma^2\mu\mu' - 2r\nu\mu'^2 + r\mu\nu\mu''}{\mu\nu'} > 0\,, \tag{27}$$

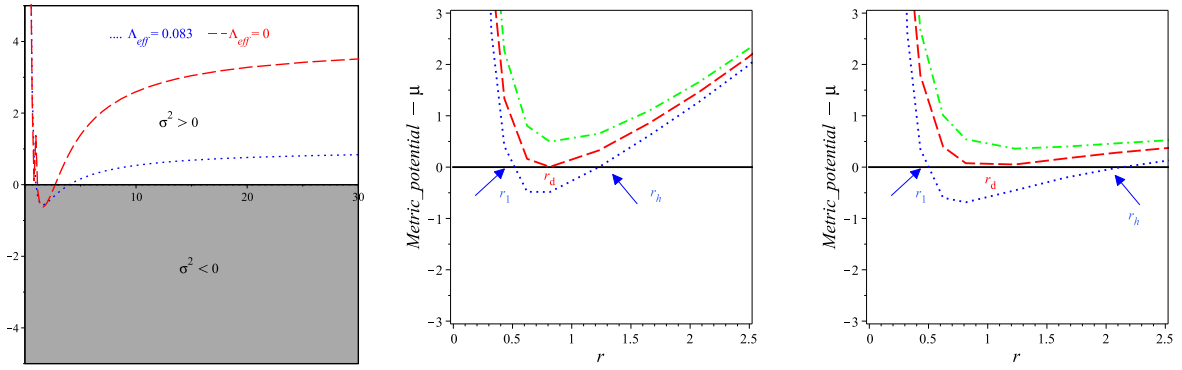
where μ and ν are given by Eq. (19). Using Eq. (27) one can get the following form of σ^2 as:

$$\sigma^2 = \frac{3\mu\nu\mu'' - 2r\nu\mu'^2 + r\mu\nu\mu''}{\mu^2\nu'^2} > 0\,. \tag{28}$$

Eq. (28) is plotted in Fig. 1 using specific values of the model. In this figure we study $\Lambda_{eff} = 0$, Reissner-Nordström GR spacetime and $\Lambda_{eff} \neq 0$ of the BH solution (19). The two cases display the regions where BH solution is stable/unstable by unshaded and shaded regions, respectively.

and

Equation



(a) Stability of the BH for the case $\Lambda_{eff}=0$ and $\Lambda_{eff.}=0.083$

(b) Horizons of the linear Maxwell field

(c) Horizons of the non–linear Maxwell field

Figure 1. Plot (a) shows the behavior of Eq. (28) viz r for BH (19). The behavior of the metric potential $\mu(r)$ which characterizes the horizons by putting $\mu(r)=0$: (b) for linear Maxwell Rastall gravity theory; (c) for the nonlinear electrodynamics Rastall's theory. The values of m for the linear case are 1.3; 0.99; 0.8 and q=1 while for the nonlinear case m=1.3; 1.1 and 0.9, q=1 and $\Lambda_{eff.}$=0.3

III. THE THERMODYNAMICAL PROPERTIES OF THE BH (19)

The thermodynamics of BH is considered an interesting topic in physics because it enables us to understand the physics of the solution. Two main approaches have been proposed to understand the thermodynamical quantities of the BHs: The first approach delivered by Gibbons and Hawking [57, 58] constructed to understand the thermal properties of the Schwarzschild BH through the use of Euclidean continuation. In the second approach, one has to define the gravitational surface from where we can define Hawking temperature then one can able to study the stability of the BH [59–62]. Here, we are going to follow the second approach to investigate the thermodynamics of the (A)dS BH obtained in Eq. (19) and then analyzed its stability. The physical quantities characterized by the BH (19) are the mass, m, the charge, and the effective cosmological constant $\Lambda_{eff.}$.

The horizons of Eq. (19), is calculated by deriving the roots of $\mu(r)=0$ which we plot it in Figs. 1(b) and 1(c) using specific values. Plots of Figs. 1(b) and 1(c) indicate the roots of $\mu(r)$ that fix the horizons of BH (19), i.e., r_1 and r_h . We should emphasize that, the linear case, for $m>0$, $q>0$ and $\Lambda_{eff.}=0$, we can show that the two roots can be formed when $m>m_{min}>q$. However, when $m=m_{min}$, we fix the degenerate horizons, i.e., r_{dg}, at which $r_1=r_h$, which is the Nariai BH whose thermodynamics is studied [63–65]. However, when $m<m_{min}<q$, there is no BH formed which means that we have a naked singularity as shown in Fig. 1(b). Same discussion can be used to the NED case, where the degenerate horizon is shown in Fig. 1(c) [64–71]. In this study, we use positive values of effective cosmological constant, because these give two horizons. Nevertheless, it is important to mention that negative values of the effective cosmological constant creates same pattern which is characterized by two horizons [72, 73].

The stability of the BH depends on the sign of the heat capacity C_h. Now, we are going to discuss the thermal stability of the BHs through their behavior of heat capacities [10, 74–76]:

$$H_c=\frac{dE_h}{dT_h}=\frac{\partial m}{\partial r_h}\left(\frac{\partial T}{\partial r_h}\right)^{-1}, \tag{29}$$

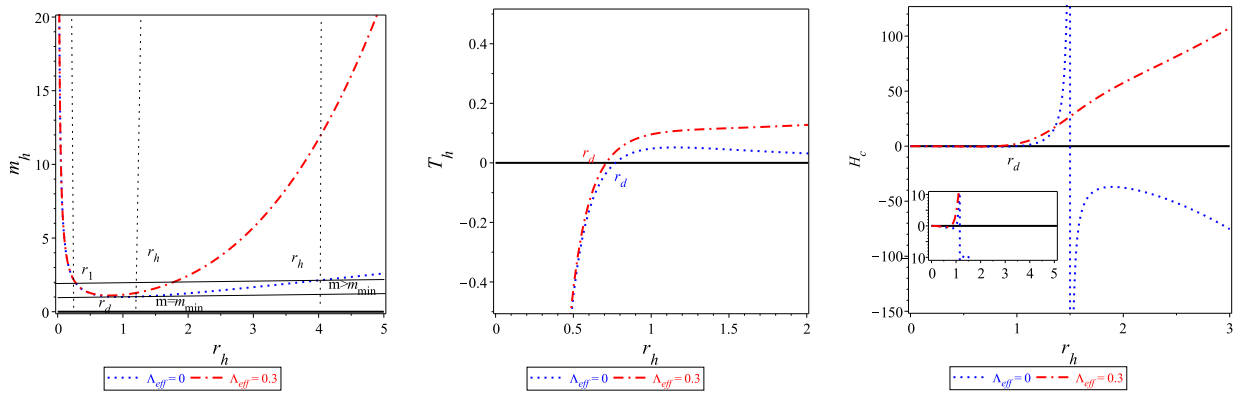
where E_h is the energy. If $H_c>0$ or $(H_c<0)$, the BH will thermodynamically stable/ unstable, respectively. To understand this process, we suppose that at some point the BH absorbs more radiation than it emits, which yields positive heat capacity which means that the mass is indefinitely increases. Contrary, when the BH emits more radiation than it absorbs, this yields negative heat capacity which means the BH mass is indefinitely decreases until it disappears. Therefore, BH that has negative heat capacity is unstable thermally.

To calculate Eq. (29), we need the analytical forms of $m_h\equiv m(r_h)$ and $T_h\equiv T(r_h)$. Therefore, let us calculate the mass of the BH in an event horizon r_h. Therefore, we put $\mu(r_h)=0$, which yields:

$$m_h \overset{Eq.(19)}{=} \frac{\Lambda_{eff} {r_h}^4+{r_h}^2+q^2}{2r_h}. \tag{30}$$

Eq. (30) shows that the total mass of BH is function of r_h, the charge and $\Lambda_{eff.}$. For specific value of the charge we

Equation



(a) Mass–radius relation (b) Hawking temperature–radius relation (c) Heat capacity–radius relation

Figure 2. Plots of thermodynamical quantities of BHs: (a) The mass-radius relation which determines the minimal mass; (b) The hawking temperature which vanishes at r_h; (c) The heat capacity. Moreover, the linear case investigates a second-order phase transition. All the figures are plotted for $m_h = q = 1$.

plot the relation of the horizon mass-radius in Fig. 2(a) which shows:

$$m(r_h \to 0) \to \infty, \qquad m(r_h \to \infty) \to \infty. \tag{31}$$

The temperature of BH is calculated at the outer event horizon $r = r_h$ as [77]:

$$T = \frac{\kappa}{2\pi}\,. \tag{32}$$

Here κ is the surface gravity defined as $\kappa = \frac{\mu'(r_h)}{2}$. The temperatures of the BH (18) is given by:

$$T_h \overset{Eq.(18)}{=} \frac{1}{2\pi}\left(\Lambda_{eff} r_h + \frac{1}{{r_h}^2}\left[m - \frac{q^2}{r_h}\right]\right), \tag{33}$$

with T_h being the temperature at r_h. For our two cases, linear and nonlinear electrodynamics, we depict the temperatures in Fig. 2(b) for specific values. Figure 2(b) shows that the horizon temperature T_h has a zero value at $r_h = r_{dg}$. However, when $r_h < r_{dg}$, the horizon temperature becomes negative and forms an ultracold black hole. This result was discussed by Davies [78] who said that there are no obvious reasons from the thermodynamical viewpoint that prevents a BH temperature to become negative and linked this to a naked singularity. Actually, this is what happened in Fig. 2(b) when $r_h < r_{min}$ region. The case of ultracold BH is explained in the existence of phantom energy field [79], which investigates decreasing of the mass behavior in Fig. 2(a). When $r_h > r_{dg}$, the temperature become positive. When r_h becomes larger, the temperatures of both cases, linear and nonlinear pattern in a similar manner.

Now we are going to evaluate the heat capacity, H_c. Using Eqs. (29), (30) and (33) we get:

$$H_c \overset{Eq.(18)}{=} \frac{2\pi {r_h}^2\left[3\Lambda_{eff}{r_h}^4 + {r_h}^2 - q^2\right]}{2(\Lambda_{eff}{r_h}^4 + 3q^2 - 2mr_h)}\,. \tag{34}$$

The above equation is not easy to get from it any information, thus we depicted it in Fig. 2(c) using specific values of the parameters. As Fig. 2(c) shows that in the both cases of linear and nonlinear charged BH solutions, H_c vanishes at r_{dg} and also their temperatures. In GR limit, the linear case, H_c has positive values when $r_h > r_{dg}$, however, when $r_h < r_{dg}$ it has negative values. In the NED case, the heat capacity is always positive unless $r_h < r_{dg}$.

A. First law of thermodynamics of the BH solution (18)

Using Eq. (30) we get:

$$M = m_h = \frac{\Lambda_{eff}{r_h}^3}{2} + \frac{r_h}{2} + \frac{q^2}{2r_h}\,. \tag{35}$$

Moreover, from the definition of entropy:

$$S = \frac{A}{4} = \pi {r_h}^2 , \tag{36}$$

we can show that the effective cosmological constant and pressure are given as [80]:

$$P = \frac{3\Lambda_{eff}}{8\pi} . \tag{37}$$

Eq. (35) can be rewritten in terms of pressure and entropy as:

$$M(S, q, P) = \frac{1}{6\sqrt{\pi S}} \left(3\pi\, q^2 + 3S + 8P\, S^2\right) . \tag{38}$$

Therefore, the parameters related to S, q and P are calculated as:

$$\begin{aligned} T &= \left(\frac{\partial M}{\partial S}\right)_{P,q} = \frac{1}{4\pi r_h}\left(1 - \frac{q^2}{{r_h}^2} + 3\pi {r_h}^2 \Lambda_{eff.}\right) , \\ \xi &= \left(\frac{\partial M}{\partial q}\right)_{S,P} = \frac{q}{r_h} , \qquad V = \left(\frac{\partial M}{\partial P}\right)_{S,q} = \frac{4}{3}\pi {r_h}^3 , \end{aligned} \tag{39}$$

with ξ, T, and V are the electric potential, temperature, and thermodynamic volume, respectively. Using the above equations we can get the following Smarr relation

$$M = 2TS + \xi q - 2VP , \tag{40}$$

from which it is easy to prove the first law of thermodynamics as:

$$dM = TdS + \xi\, dq + V dP . \tag{41}$$

Equation (40) ensures the validity of the first law of the BH (19).

IV. DISCUSSION AND CONCLUSIONS

In this research, we have considered spherically symmetric BH in Rastall's theory of gravity. We study the NED spherically symmetric spacetime and derived an exact solution that is affected by the Rastall parameter. This is the first time to derive a NED BH solution from the field equation of Rastall's gravitational theory. The main contribution of Rastall's parameter in this study comes from the contribution of the trace of the NED which has a non-vanishing value in contrast to the linear Maxwell theory. We show that the effect of the Rastall parameter acts as a cosmological constant and the BH behaves asymptotically as (A)dS Reissner-Nordström spacetime. When the Rastall parameter vanishes we get spacetime which asymptotes as flat Reissner-Nordström spacetime.

We have used the geodesic deviation to obtain the stability of the geodesic motion of the NED case. Moreover, we studied the horizons and showed that the BHs presented in this study could have two horizons: one is the event horizon r_1, and the other is the effective cosmological one r_h. Also, we fixed the minimum value of the BH mass that occurred at the degenerate horizon. Also, we have studied the thermal phase transitions and showed, in the linear electrodynamics case, i,e. $\epsilon = 0$, the temperature became negative when $r_h < r_d$ and therefore, heat capacity became negative and thus we have unstable BH [81–84]. Same conclusions can be applied to the NED case, however, at $r_h > r_d$, we have a positive value of the H_c which yields a stable BH. Finally, we proved the validity of the first law of thermodynamics.

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