September, 4th. 2022

Manuscript number: universe-1882952
Title: Nonlinear charged black hole solution in Rastall gravity
Author: G. G. L. Nashed

Dear Professor Chandler Qiao, Editor of Universe,

My response to the referees' report for the aforementioned manuscript is attached below. I revised the manuscript in response to all of the referees' comments. All modifications in the manuscript appear as red-colored texts.
Dear Referees,
Thank you very much for your careful reading of the manuscript and your helpful suggestions and comments. I am very sorry because the previous manuscript made you confused. Ihave revised the manuscript to take all the points in your report into consideration.

Sincerely,
Author

Reply to Reviewer 1

- **Referee**:
  As a minor objection, this referee has a query that the author might want to address before publication. The paper "Rastall gravity is equivalent to Einstein gravity",

  by Matt Visser, arXiv:1711.11500 [gr-qc], leaves this referee wondering if the results presented here could be derived more simply by resorting to said equivalence. A brief explanation (to what extent might the equivalence between Einstein and Rastall gravity affect the results of the paper under review?) would suffice.

  Author:
  I agree with the referee that the results presented in this study are equivalent to Einstein's gravity plus an arbitrary cosmological constant, which is equivalent to the result presented in 1711.11500 [gr-qc]. I have added this as a footnote on page 6 and state:

  This result is consistent with what have done in [**?**] where the author has shown that the Rastall theory is equivalent to Einstein's general relativity or equivalent to Einstein's field equation plus an arbitrary cosmological constant.

Reply to Reviewer 2

- **Referee**:
  However, as can be seen in the attached part of the report, there are a number of typographical errors (which I am sure will find many more if examined in detail). They must be fixed.

  Author:
  I am very grateful to the Referee for the valuable time she/he spent improving my study. I have followed all your valuable comments and I have reviewed carefully the manuscript and fixed the English language as much as I can.

- **Referee**:
  I see that applications (in the literature) related to Rastall gravity are not mentioned enough. This reduces the importance of the subject. My recommendation is to include the articles seen below:
  1. Shadow cast of non-commutative black holes in Rastall gravity Modern Physics Letters A 2050163 2020
  2. Quasinormal Modes of Black Holes with Non-Linear-Electrodynamic sources in Rastall Gravity Physics of the Dark Universe, 100860 (2021)
  3. Dirac quasinormal modes of power-Maxwell charged black holes in Rastall gravity 2005.00674v1
  4. Quasinormal Modes of Non-Linearly Charged Black Holes surrounded by a Cloud of Strings in Rastall Gravity arXiv: 2111.00854v1
  5. Quasinormal mode and spectroscopy of a Schwarzschild black hole surrounded by a cloud of strings in Rastall gravity Phys. Rev. D 101, 104023 (2020)
  6. Black hole solutions and Euler equation in Rastall and generalized Rastall theories of gravity Mod. Phys. Lett. A, 34 1950304 (2019)

7. Rastall cosmology Int. J. Mod. Phys. Conf. Ser., 18, 67-76 (2012)
8. GUP Modified Hawking Radiation in Bumblebee Gravity Nuclear Physics B Volume 946, September 2019, 114703
9. Gravitational Collapse in Rastall Gravity Physics Letters B 793 (2019) 276
10. Neutron Stars in Rastall Gravity Phys. Rev. D, 92 (2015)

Author:
I have followed the comments of the Referee and have discussed and added all the above valuable citations. Please see references [29–36] and [38–39].

- **Referee**:
Some statements about the future of this study should also be included in the Conclusion section
Author:
I have added some comments on the future work and state:
In this study, we have discussed Rastall's theory using a special form of non-linear electrodynamics. However, a deeper analysis is necessary, possibly regarding quantum effects in the universe. Meanwhile, the effects of Rastalls cosmology on the formation and properties of non-linear structures is a very promising research program. Furthermore, the study of $f(R)$-Rastall's theory will be extremely rich in the context of astrophysics [65]. Within the frame of $f(R)$, a BH which is similar to Reissner-Nordström BH is presented [102] for a specific form of $f(R)$. Is it possible to derive a similar solution within Rastall's $f(R)$? This study will be carried out elsewhere.
Please see the last paragraph in final section.

- **Referee**:
Finally, according to my point of view, this article needs Moderate Revision.
Author:
I thank the Referee for his comment that indeed improves the presentation. I have checked all the contents of the paper and revised it as much as I could to meet the merits of publication in this journal. Please see the revised version.

Reply to Reviewer 3

- **Referee**:
Would the solution (18) be a general spherically symmetric solution in this system? Why do the effects by the Rastall gravity and the nonlinear electrodynamics vanish simultaneously in the $c_3 = 0$ limit? Then the author may discuss some assumptions to obtain an another solution that would not coincide with the Reissner-Nordström-(A)dS one.
Author:
The solution presented by Eq. (18) is not a general spherically symmetric solution for the system of differential Eqs. (17). It is only an exact solution to such a system. The main reason for disappearing the Rastall parameter and the nonlinear electrodynamics when $c_3 = 0$ is the fact that the arbitrary function, in that case, will coincide with GR, and thus Rastall theory will coincide with GR as the solution of that case is the Riessner-Nordström solution because the trace of the electromagnetic solution will be vanishing. We have cleared this after Eqs. (17) and (18) in the revised version. In the frame of spherically symmetric spacetime and by using the charged field equation of Rstall's theory, we can not find any solution that differs from Riessner-Nordström (A)dS solution. This may be true to find a solution different from Riessner-Nordström (A)dS solution when one deals with a generalization of Rastall's theory, i.e., f(R)-Rastall's theory. We have stated this clearly after Eq. (18).

- **Referee**:
Since the obtained solution is the Reissner-Nordstrom-(A)dS one, there would be some overlaps between the stability analysis in the sections II-C and III in the present manuscript and the papers arXiv:1902.06783 and arXiv:hep-th/9908022. Then the author may modify these sections to clear the difference between the present discussions and these previous works.
Author:
I thank the Referee very much for this challenging comment. The spacetime derived in arXiv:1902.06783 is not Reissner-Nordstrom-(A)dS because the asymptote behavior of the solution presented in arXiv:1902.06783 is not a flat spacetime due to the contribution of the specific form of $f(R)$ used in this study. We have cleared this in the final paragraph of the final section. Also, we have discussed overlaps between the thermodynamics of the

present study and the one presented in hep-th/9908022. Please see the last sentence in the first paragraph on page 10.

- **Referee**:
  The terms $d\phi^2$ in the equation (14) and $\sin^2 d\phi^2$ below the equation (21) would be replaced with the terms $\sin^2\theta d\phi^2$ and $\sin^2\theta d\phi^2$, respectively.
  Author:
  I appreciate the careful reading of the Referee. Please see the revised version where your valuable comments are taken into account.

  I hope that the revised version meets the comments of the Referee to merit publication in the journal of the universe.

Sincerely yours,
Author